COMMENT

Squeezed coherent states of the hydrogen atom

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Abstract. The O(4) algebra of the hydrogen atom is made use of to define its squeezed coherent states. It is shown that the effect of squeezing is to increase the energy of the atom in its levels. The corrections to the Bohr formula and the spectral transitions due to squeezing are calculated.

Recently Gerry (1984) introduced oscillator-like coherent states on the O(4) algebra of the hydrogen atom, and showed that the correct classical limit could be obtained without the correspondence limit. He also indicated that the coherent states thus introduced describe the 'elliptical orbits' anticipated in 1926 by Schrödinger (1978). More recently Bhaumik *et al* (1986) using these coherent states constructed a wavepacket which travels on an elliptic trajectory. In fact all these works are further developments on the realisation of the connection between the hydrogen atom and a four-dimensional oscillator with a constraint which has been rediscovered by many authors (Kibler and Negadi 1983, Cornish 1984, Chen and Kibler 1985) since the original discovery by Pauli (see the elaborate review, incidentally the first such one, by Bander and Itzykson (1966)). Also Nieto (1980) obtained coherent states for the Coulomb potential which are different from those introduced by Gerry (1984) and Bhaumik *et al* (1986). Nieto developed a formalism to obtain coherent states for general potentials using Schrödinger's criterion, i.e. to define coherent states as those with undistorted wavepackets with classical motion. Coherent states of Gerry (1984) and Bhaumik *et al* (1986) are the minimum uncertainty states of the harmonic oscillator (see Klauder and Sudarshan 1966). In this comment we introduce the oscillator squeezed coherent states on the O(4) algebra of the hydrogen atom. Further the effects of 'squeezing' are calculated.

First we shall briefly discuss squeezed coherent states and their importance. Consider the harmonic oscillator Hamiltonian

$$H = \hbar\omega(a^{\dagger}a + \tfrac{1}{2}) \tag{1}$$

with $[a, a^{\dagger}] = 1$. The eigenstates of H are given by $|n\rangle$ where $a^{\dagger}a|n\rangle = n|n\rangle$.

The coherent state $|\alpha\rangle$ is obtained as

$$|\alpha\rangle \equiv D(\alpha)|0\rangle \tag{2}$$

where $D(\alpha)$ is the displacement operator given by

$$D(\alpha) = \exp(\alpha a^{\dagger} - \alpha^{*}a) \tag{3}$$

and α a complex number.

Squeezed coherent states (scs) are defined as

$$|\alpha, Z\rangle \equiv D(\alpha)S(Z)|0\rangle \tag{4}$$

where $S(Z)$ is the squeezing operator given by

$$S(Z) = \exp[\tfrac{1}{2}Za^{\dagger}a^{\dagger} - \tfrac{1}{2}Z^{*}aa]. \tag{5}$$

Z is known as the squeezing parameter.

The squeezing operator $S(Z)$ transforms a and $a^{\dagger}$ into another equivalent bosonic system b and $b^{\dagger}$ as

$$SaS^{\dagger} = a \cosh r + e^{i\theta}a^{\dagger} \sinh r = b$$
$$Sa^{\dagger}S^{\dagger} = a^{\dagger} \cosh r + e^{-i\theta}a \sinh r = b^{\dagger} \tag{6}$$

where $Z = r\,e^{i\theta}$.

The scs have the following two remarkable properties.

(1) *The classical motion of the wavepackets.* It is easy to see that the scs correspond to Gaussian wavepackets with widths distorted from that of the vacuum state and those states also follow classical motion but only the uncertainties oscillate (see Fisher *et al* 1984). Because of the oscillatory motion of the uncertainties Fujiwara and Wergeland (1984) named it the 'jester'.

(2) *The non-classical nature of squeezing.* Writing $a = x_1 + ix_2$ and calculating variances $V(x_1)$ and $V(x_2)$ in the scs $|\alpha, Z\rangle$ one obtains

$$V(x_1) = \tfrac{1}{4}e^{-2r} \qquad V(x_2) = \tfrac{1}{4}e^{2r}. \tag{7}$$

It has been pointed out by Walls (1983) that the condition $V(x_1) < \tfrac{1}{4}$ implies that the Glauber–Sudarshan representation P function should be a non-positive definite function. Thus squeezing is a non-classical property of the electromagnetic quanta.

Originally scs were obtained by Stoler (1970, 1971), who showed that the states with minimum uncertainty form an equivalence class, which are the scs.

There has been some activity on the quantum mechanical aspects of the squeezing operator (Fisher *et al* 1984, Santhanam and Satyanarayana 1984). The present author also introduced generalised squeezed coherent states for the harmonic oscillator (Satyanarayana 1985). (For reviews on scs see Nieto (1984) and Walls (1983).)

From (1) we obtain

$$H_{sq}^{cl} = \langle \alpha, Z|H|\alpha, Z\rangle = \hbar\omega(|\alpha|^2 + \sinh^2 Z + \tfrac{1}{2}). \tag{8}$$

Now we can define the action variables as

$$J = h(|\alpha|^2 + \sinh^2 Z + \tfrac{1}{2}) \tag{9}$$

and

$$H_{sq}^{cl} = \nu J.$$

Also

$$\nu = \partial H_{sq}^{cl}/\partial J. \tag{10}$$

The quantised energy levels are recovered by invoking the Bohr–Sommerfeld rule

$$J = h(n + \sinh^2 Z + \tfrac{1}{2}). \tag{11}$$

The above form of quantisation condition is chosen in view of the fact that H_{sq}^{cl} becomes $\hbar\omega(n + \tfrac{1}{2})$ as the squeezing parameter Z approaches zero.

We shall now proceed to discuss the scs of the hydrogen atom. We employ the notation of Gerry (1984).

The Hamiltonian of the hydrogen atom is given by

$$H = p^2/2\mu - Ze^2/r. \tag{12}$$

It is very well known that the angular momentum vector L and Pauli-Runge-Lenz vector A' given by

$$A' = -Ze^2 r/r + (1/2\mu)(L \times p - p \times L) \tag{13}$$

commute with H. Also A' is orthogonal to L and A'^2 has a term containing H. Using the decomposition $O(4) = SU(2)_a \times SU(2)_b$, Schwinger's boson realisation of $SU(2)$ and the properties of A' Gerry (1984) obtained

$$(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)^2 = (b_1^\dagger b_1 + b_2^\dagger b_2 + 1)^2 \tag{14}$$

and

$$-\mu Z^2 e^4/\hbar^2 E = (a_1^\dagger a_1 + a_2^\dagger a_2 + 1)^2 + (b_1^\dagger b_1 + b_2^\dagger b_2 + 1)^2 \tag{15}$$

where $[a_i, a_i^\dagger] = 1$ and $[b_i, b_i^\dagger] = 1$ for $i = 1, 2$.

Now introducing the Fock space basis, i.e. eigenstates of $a_i^\dagger a_i$ as $|n_i\rangle$ and $b_i^\dagger b_i$ as $|m_i\rangle$ where $i = 1, 2$, equations (14) and (15) become

$$n_1 + n_2 = m_1 + m_2 \tag{16}$$

and

$$E_n = -\mu Z^2 e^4/2\hbar^2 n^2 \tag{17}$$

where

$$n = n_1 + n_2 + 1 = m_1 + m_2 + 1.$$

We now introduce the scs as

$$\begin{aligned}|\alpha_i^a, Z_i^a\rangle &\equiv D(\alpha_i^a)S(Z_i^a)|n_i = 0\rangle \\ |\alpha_i^b, Z_i^b\rangle &\equiv D(\alpha_i^b)S(Z_i^b)|m_i = 0\rangle \end{aligned} \tag{18}$$

for $i = 1, 2$ and $Z_i^a = r_i^a$ and $Z_i^b = r_i^b$.

The condition in equation (16) reads

$$|\alpha_1^a|^2 + |\alpha_2^a|^2 + \sinh^2 r_1^a + \sinh^2 r_2^a = |\alpha_1^b|^2 + |\alpha_2^b|^2 + \sinh^2 r_1^b + \sinh^2 r_2^b. \tag{19}$$

Now, for the squeezing parameters r_i^a and r_i^b for $i = 1, 2$ tending to zero, the above equation becomes equation (17) of Gerry (1984).

In the spirit of (8), the energy becomes

$$H_{sq}^{cl} = \frac{-\mu Z^2 e^4}{2\hbar^2(|\alpha_1^a|^2 + |\alpha_2^a|^2 + \sinh^2 r_1^a + \sinh^2 r_2^a + 1)^2}. \tag{20}$$

For $r_1^a = r_2^a = 0$, the above equation becomes equation (18) of Gerry (1984).

Further we see that squeezing does not affect the relation for the period of the Kepler orbit.

Using equation (11) and taking $r_1^a = r_1$ and $r_2^a = r_2$, we obtain

$$E_{sq}^{(n)} = \frac{-\mu Z^2 e^4}{2\hbar^2(n_1 + n_2 + \sinh^2 r_1 + \sinh^2 r_2 + 1)^2}. \tag{21}$$

Equation (21) gives the energy levels of a hydrogen atom specified by the principal quantum number $n(=n_1+n_2+1)$ and the squeezing parameters r_1 and r_2. It is to be noted that the energy levels acquire higher values, as happens for the harmonic oscillator.

Taking $r_1 = r_2 = 0$, equation (21) becomes

$$E^{(n)} = -\mu Z^2 e^4 / 2\hbar^2 (n_1+n_2+1)^2 \tag{22}$$

which is equation (15) of Gerry (1984).

Now we proceed to calculate the absolute shift in energy $E_{sq}^{(n)} - E^{(n)}$:

$$E_{sq}^{(n)} - E^{(n)} = \frac{\mu Z^2 e^4}{2\hbar^2} \frac{(\sinh^2 r_1 + \sinh^2 r_2)}{(n_1+n_2+1)^2}$$
$$\times \left(\frac{(2n_1+2n_2+2+\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1+\sinh^2 r_1+\sinh^2 r_2)^2} \right). \tag{23}$$

For $\sinh^2 r_1 \ll n_1$ and $\sinh^2 r_2 \ll n_2$, we have

$$E_{sq}^{(n)} - E^{(n)} = \frac{\mu Z^2 e^4}{2\hbar^2} \frac{(\sinh^2 r_1 + \sinh^2 r_2)}{(n_1+n_2+1)^4}(2n_1+2n_2+2+\sinh^2 r_1+\sinh^2 r_2)$$
$$\times \left(1 - \frac{2(\sinh^2 r_1+\sinh^2 r_2)}{(n_1+n_2+1)} \right)$$
$$= \frac{\mu Z^2 e^4}{2\hbar^2} \frac{(\sinh^2 r_1 + \sinh^2 r_2)}{(n_1+n_2+1)^5}[n_1+n_2+1-2(\sinh^2 r_1+\sinh^2 r_2)]$$
$$\times [2(n_1+n_2+1)+(\sinh^2 r_1+\sinh^2 r_2)]. \tag{24}$$

The relative shift in energy is given by

$$\frac{E_{sq}^{(n)} - E^{(n)}}{E^{(n)}} = \frac{-(\sinh^2 r_1 + \sinh^2 r_2)[2(n_1+n_2+1)+(\sinh^2 r_1+\sinh^2 r_2)]}{(n_1+n_2+1+\sinh^2 r_1+\sinh^2 r_2)^2}. \tag{25}$$

For $\sinh^2 r_1 \ll n_1$ and $\sinh^2 r_2 \ll n_2$

$$\frac{E_{sq}^{(n)} - E^{(n)}}{E^{(n)}} = \frac{-(\sinh^2 r_1 + \sinh^2 r_2)}{(n_1+n_2+1)^3}[2(n_1+n_2+1)+\sinh^2 r_1+\sinh^2 r_2]$$
$$\times [n_1+n_2+1-2(\sinh^2 r_1+\sinh^2 r_2)]. \tag{26}$$

The spectral transition energy $\Delta E(n, r_1, r_2 \rightarrow m, r_1, r_2)$ is given below:

$$\Delta E_{(r_1,r_2)}^{(n,m)} = \frac{\mu Z^2 e^4}{2\hbar^2}[(n_1+m_1)+(n_2+m_2)+2(\sinh^2 r_1+\sinh^2 r_2+1)][(n_1-m_1)+(n_2-m_2)]$$

$$\times \frac{1}{(n_1+n_2+\sinh^2 r_1+\sinh^2 r_2+1)(m_1+m_2+\sinh^2 r_1+\sinh^2 r_2+1)}. \tag{27}$$

Equation (23) gives the squeezing correction to the spectral transitions, which arise mainly due to the quantum mechanical nature of squeezing.

In a future communication we shall investigate the nature of the motion of the wavepacket constructed using these squeezed coherent states.

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